CFD6 # Computer Fluid Dynamics

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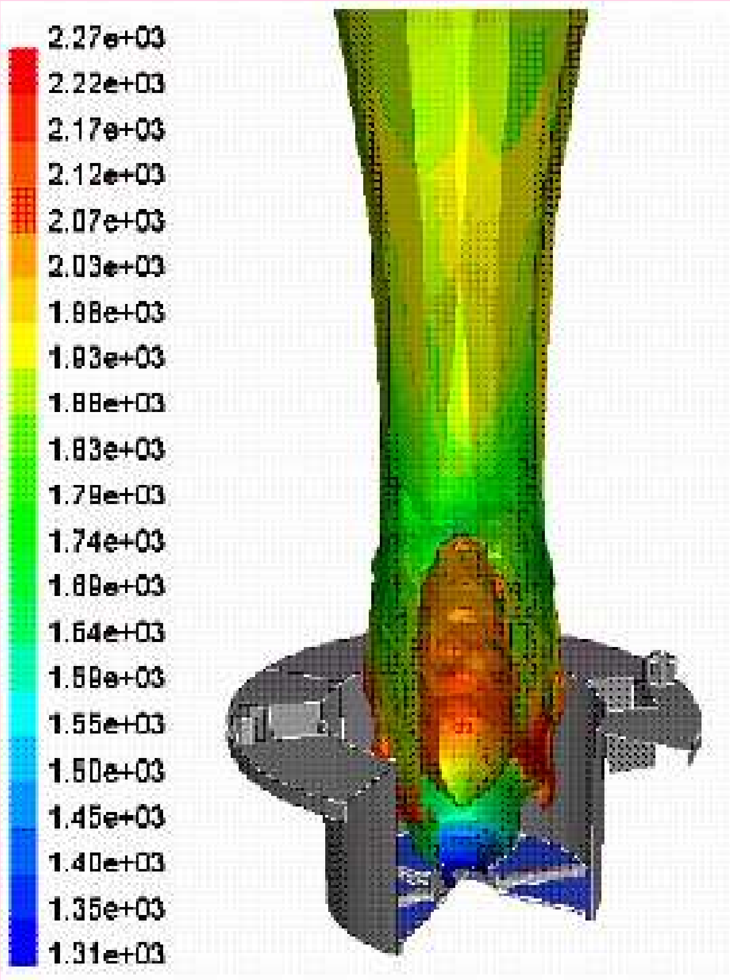

Turbulent flows,…

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses

Rudolf Žitný, Ústav procesní a zpracovatelské techniky ČVUT FS 2010

Some figures and equations are copied from Wikipedia

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Turbulence

Source of nonlinearities of NS equations are inertial and viscous terms

$$\rho(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}) = -\nabla p + \nabla \cdot (\mu(\vec{e}) \nabla \vec{u}) + \vec{S}_M$$

$$\rho(\frac{\partial \vec{u}}{\partial t} + \nabla \bullet (\vec{u}\vec{u}) - \vec{u} \nabla \bullet \vec{u}) = -\nabla p + \nabla \cdot (\mu(\vec{e}) \nabla \vec{u}) + \vec{S}_M$$

for incompressible fluids the equivalent **conservative form**

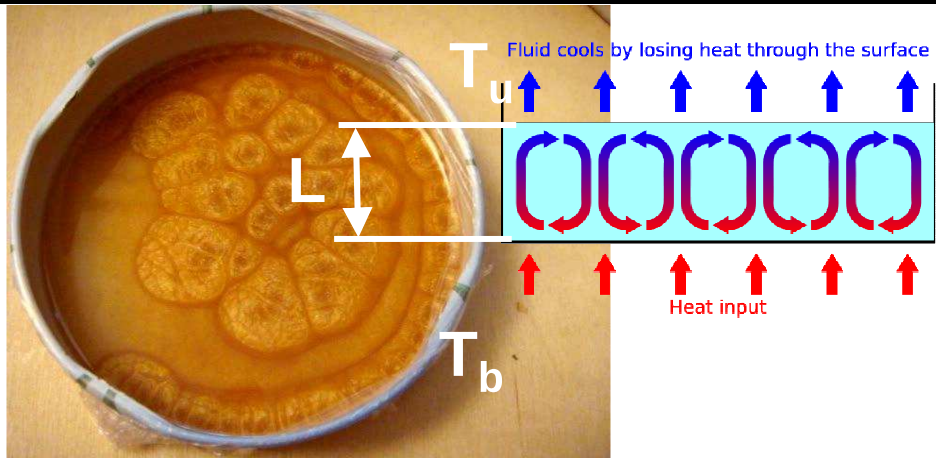
$$\rho(\frac{\partial \vec{u}}{\partial t} + \nabla \bullet (\vec{u}\vec{u})) = -\nabla p + \nabla \cdot (\mu(\vec{e}) \nabla \vec{u}) + \vec{S}_M$$

The convection term is a quadratic function of velocities – source of nonlinearities and turbulent phenomena

CFD6 Instabilities Rayleigh Benard convection

Horizontal liquid layer heated from below is in still (viscous forces attenuate small disturbances) until the buoyancy exceeds a critical level Ra_L. Then the more or less regular cells with circulating fluid are formed.

$$Ra_L = \frac{g\beta}{\nu\alpha}(T_u - T_b)L^3 \quad \mathbf{>1100}$$

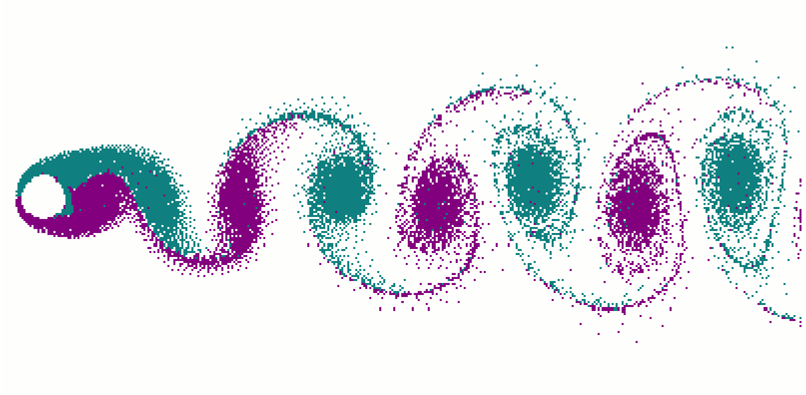


Even if the stability limit was exceeded the flow pattern is steady and remains laminar. At this stage the nonlinear convective term is not so important. Only if $Ra_L > 10^9$ (approximately) eddies start to be chaotic, velocity fluctuates and the flow is turbulent.

CFD6 Instabilities Karman vortex street

Repeating pattern of swirling vortices caused by the unsteady separation of flow of a fluid over bluff bodies. A vortex street will only be observed above a limiting Re value of about 90.

$$\frac{fd}{V} = 0.198\left(1 - \frac{19.7}{Re}\right)$$



Even if the stability limit was exceeded the flow pattern is steady and remains laminar.

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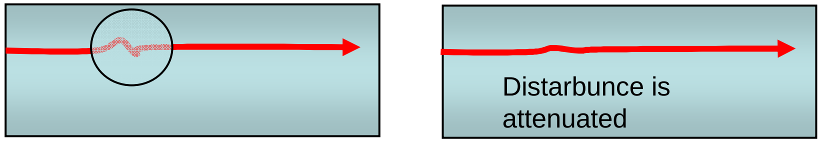
Turbulence

Relative magnitude of inertial and viscous terms is Reynolds number

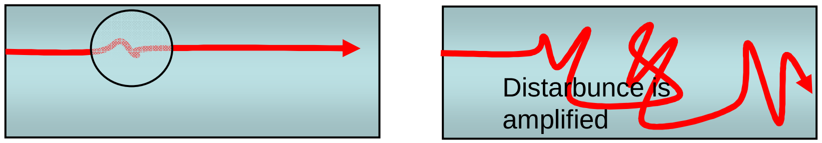
$$\mathrm{Re} = \frac{\rho \nabla \bullet (\vec{u}\vec{u})}{\mu \nabla^2 \vec{u}} = \frac{\rho u D}{\mu}$$

Increasing Re increases nonlinearity of NS equations. This nonlinearity leads to sensitivity of NS solution to flow disturbances.

Laminar flow: $\mathrm{Re} < \mathrm{Re}_{crit}$



Turbulent flow: $\mathrm{Re} > \mathrm{Re}_{crit}$



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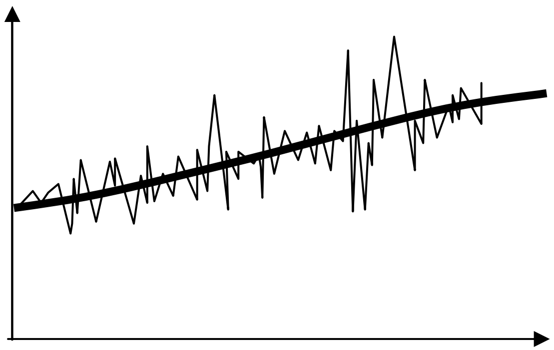
Turbulence - fluctuations

Turbulence can be defined as a deterministic chaos. Velocity and pressure fields are NON-STATIONARY (du/dt is nonzero) even if flowrate and boundary conditions are constant. Trajectory of individual particles are extremely sensitive to initial conditions (even the particles that are very close at some moment diverge apart during time evolution). .

Velocities, pressures, temperatures… are still solutions of NS and energy equations, however they are nonstationary and form chaotically oscillating vortices (eddies). Time and spatial profiles of transported properties are characterized by fluctuations



$$u = \overline{u} + u'$$

Actual value at given time and space

Mean value

fluctuation

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Turbulence - fluctuations

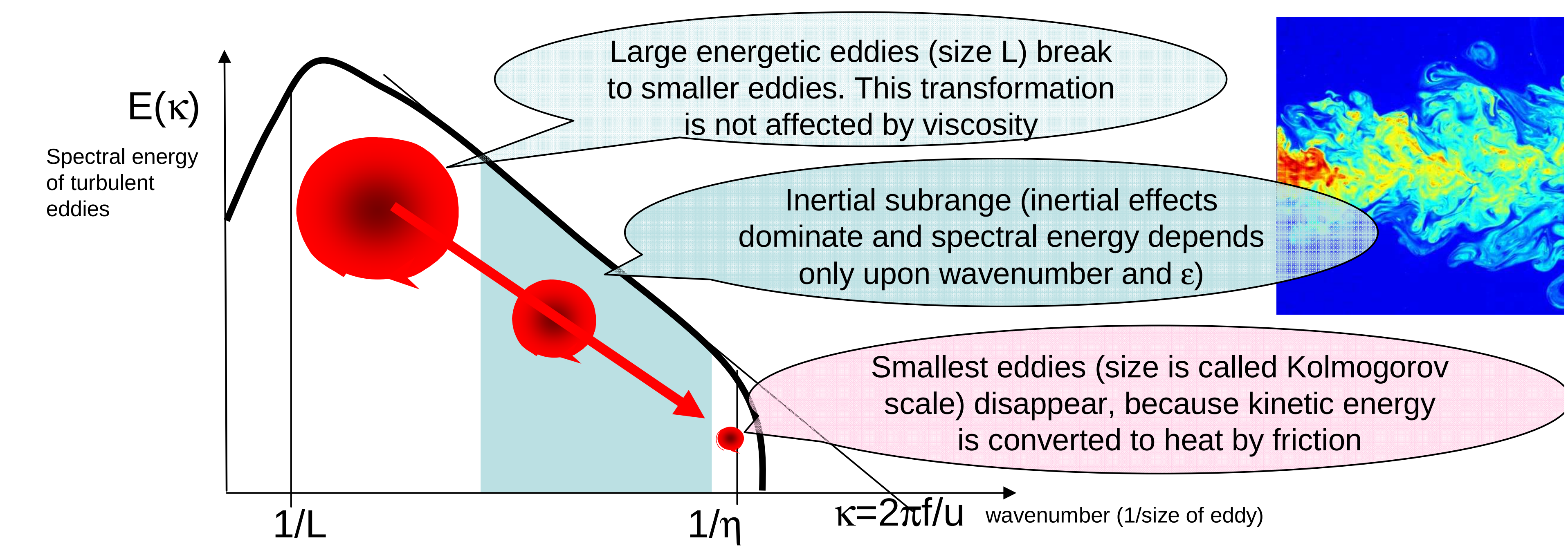
Statistics of turbulent fluctuations

- Mean values $\bar{u}, \bar{p}, \bar{\rho}, \bar{T}$ (remark: mean values of fluctuations are zero)
- rms (root mean square) $\sqrt{\overline{u'^2}}$
- Kinetic energy of turbulence $k = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$
- Intensity of turbulence $I = (\sqrt{\frac{2}{3}k})/\bar{u}$
- Reynolds stresses $\tau_{t,xx} = \rho\overline{u'^2},\ \tau_{t,yy} = \rho\overline{v'^2},\ \tau_{t,zz} = \rho\overline{w'^2},\ \tau_{t,xy} = \rho\overline{u'v'},\ \tau_{t,xz} = \rho\overline{u'w'}$

Turbulent eddies - scales

Kinetic energy of turbulent fluctuations is the sum of energies of turbulent eddies of different sizes

$$k = \frac{1}{2}\overline{u_i^{/} u_i^{/}} = \int_0^\infty E(\kappa)\, d\kappa$$



Typical values of frequency f~10 kHz, Kolmogorov scale η~0.01 up to 0.1 mm

Kolmogorov scale η decreases with the increasing Re

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Turbulent eddies - scales

Kolmogorov scales (the smallest turbulent eddies) follow from dimensional analysis, assuming that averything depends only upon the kinematic viscosity ν and upon the rate of energy supply ε in the energetic cascade (only for small isotropic eddies, of couse)

$$\eta = \sqrt[4]{\frac{\nu^3}{\varepsilon}} \qquad \tau = \sqrt{\frac{\nu}{\varepsilon}} \qquad u = \sqrt[4]{\nu\varepsilon}$$

Length scale

Time scale

velocity scale

These expressions follow from dimension of viscosity ν [m^2/s] and the rate of energy dissipation ε [m^2/s^3]

$$\eta = \nu^{\alpha}\varepsilon^{\beta}$$

$$[m] = [\frac{m^{2\alpha}}{s^{\alpha}}][\frac{m^{2\beta}}{s^{3\beta}}] = \left[\frac{m^{2\alpha+2\beta}}{s^{\alpha+3\beta}}\right]$$

$$\left.\begin{aligned} 1 &= 2\alpha + 2\beta \\ 0 &= \alpha + 3\beta \end{aligned}\right] \longrightarrow \begin{aligned} \alpha &= 3/4 \\ \beta &= -1/4 \end{aligned}$$

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Example tutorial

Derive time scale of the smallest turbulent eddies

$$\tau = \sqrt{\frac{\nu}{\varepsilon}}$$

Time scale

$$\tau = \nu^{\alpha}\varepsilon^{\beta}$$

$$[s] = [\frac{m^{2\alpha}}{s^{\alpha}}][\frac{m^{2\beta}}{s^{3\beta}}] = \left[\frac{m^{2\alpha+2\beta}}{s^{\alpha+3\beta}}\right]$$

$$\left.\begin{aligned} 0 &= 2\alpha + 2\beta \\ -1 &= \alpha + 3\beta \end{aligned}\right] \longrightarrow \begin{aligned} \alpha &= 1/2 \\ \beta &= -1/2 \end{aligned}$$

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Turbulent eddies - scales

Wikipedia (abbreviated)

Turbulent flow is composed by "eddies" of different sizes. The sizes define a characteristic length scale for the eddies, which are also characterized by velocity scales and time scales (turnover time) dependent on the length scale. The large eddies are unstable and break up originating smaller eddies, and the kinetic energy of the initial large eddy is divided into the smaller eddies that stemmed from it. The energy is passed down from the large scales of the motion to smaller scales until reaching a sufficiently small length scale such that the viscosity of the fluid can effectively dissipate the kinetic energy into internal energy.
In his original theory of 1941, Kolmogorov postulated that for very high Reynolds number, the small scale turbulent motions are statistically isotropic. In general, the large scales of a flow are not isotropic, since they are determined by the particular geometrical features of the boundaries (the size characterizing the large scales will be denoted as *L*).

Kolmogorov introduced a hypothesis: for very high Reynolds numbers the statistics of small scales are universally and uniquely determined by the kinematic viscosity (ν) and the rate of energy dissipation (ε). With only these two parameters, the unique length that can be formed by dimensional analysis is $\eta = (\nu^3/\varepsilon)^{1/4}$

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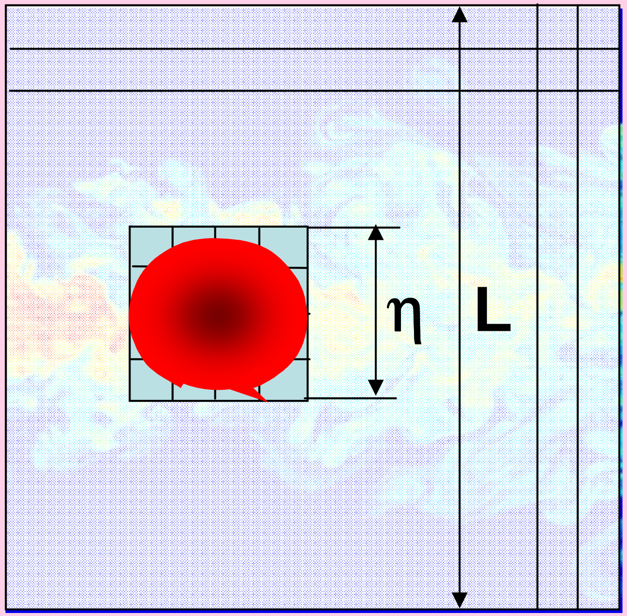
DNS Direct Numerical Simulation



In order to resolve all details of turbulent structures it is necessary to use mesh with grid size less than the size of the smallest (Kolmgorov) eddies. N-grid points in one direction should be

$$N > \frac{L}{\eta} \qquad \varepsilon \simeq \frac{u^3}{L} \text{ (based upon dimensional ground)}$$

$$N^3 > \frac{L^3}{\eta^3} = \frac{L^3 \varepsilon^{3/4}}{\nu^{9/4}} = \frac{L^3 u^{9/4}}{\nu^{9/4} L^{3/4}} = \left(\frac{Lu}{\nu}\right)^{9/4} = \mathrm{Re}_\tau^{9/4}$$

Velocity scale u in previous expression is related to magnitude of turbulent fluctuations (rms of u', or √k). The Re_τ related to the velocity fluctuation is called turbulent Reynolds number.

$\mathrm{Re} = \frac{\overline{u}L}{\nu}$	$\mathrm{Re}_\tau = \frac{uL}{\nu}$	No.of grid points in DNS	No.of time steps
12300	380	6.7 M	32 k
30800	800	40 M	47 k
61600	1450	150 M	63 k
230000	4650	2100 M	114 k

Table concerns DNS modelling of channel flow experiments (rewritten from Wilcox: Turbulence modelling, chapter 8).

$$N_{DNS} \sim \mathrm{Re}_\tau^{9/4}$$

$$N_{time\_steps} \sim \mathrm{Re}_\tau^{3/4}$$

Remark: Re~10^6 or 10^7 at flow around a car or flow around wings

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DNS Direct Numerical Simulation

Direct numerical simulation of transition and turbulence in compressible mixing layer

FU Dexun , MA Yanwen ZHANG Linbo

Vol 43 No.4, SCIENCE IN CHINA (Series A), April 2000

Abstract The three-dimensional compressible Navier-Stokes equations are approximated by a fifth order upwind compact and a sixth order symmetrical compact difference relations combined with threestage Ronge-Kutta method. The computed results are presented for convective Mach number ***Mc*** = 0.8 and Re = 200 with initial data which have equal and opposite oblique waves. From the computed results we can see the variation of coherent structures with time integration and full process of instability, formation of A -vortices, double horseshoe vortices and mushroom structures. The large structures break into small and smaller vortex structures. Finally, the movement of small structure becomes dominant, and flow field turns into turbulence. It is noted that production of small vortex structures is combined with turning of symmetrical structures to unsymmetrical ones. It is shown in the present computation that the flow field turns into turbulence directly from initial instability and there is not vortex pairing in process of transition. It means that for large convective Mach number the transition mechanism for compressible mixing layer differs from that in incompressible mixing layer.

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DNS Direct Numerical Simulation

Fluid Mechanics and Its Applications, 2004, Volume 65, 2, 99-114, DOI: 10.1007/0-306-48383-1_7

DNS AND LES OF TURBULENT BACKWARD-FACING STEP

FLOW USING 2ND- AND 4TH-ORDER DISCRETIZATION

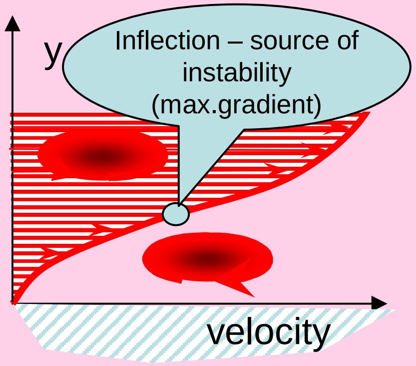
ADNAN MERI AND HANS WENGLE

Abstract. Results are presented from a Direct Numerical Simulation (DNS) and Large-Eddy Simulations (LES) of turbulent flow over a backward-facing step with a fully developed channel flow utilized as a time-dependent inflow condition. Numerical solutions using a fourth-order compact (Hermitian) scheme, which was formulated directly for a non-equidistant and staggered grid in [1] are compared with numerical solutions using the classical second-order central scheme. The results from LES (using the dynamic subgrid scale model) are evaluated against a corresponding DNS reference data set (fourth-order solution).

CFD6 Transition Laminar-Turbulent

Hydrodynamic instability due to prevailing inertial forces (convection term in NS equations) is the cause of turbulence.

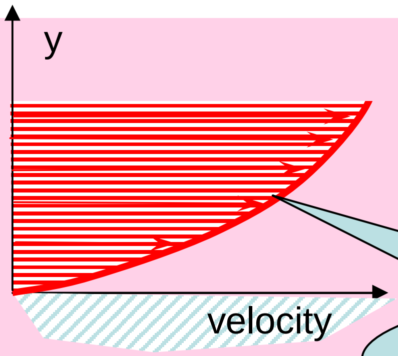
➢**Inviscid** instabilities



characterised by existence of inflection point of velocity profile

- jets
- wakes
- boundary layers wit adverse pressure gradient $\Delta p>0$

➢**Viscous** instabilities



Linear eigenvalues analysis (Orr-Sommerfeld equations)

- channels, simple shear flows (pipes)
- boundary layers with $\Delta p>0$

Poiseuille flow ~ 5700
Couette flow – stable?

There is **no** inflection of velocity profile in a pipe, however turbulent regime exists if Re>2100

CFD6 Transition Laminar-Turbulent

How to indentify whether the flow is laminar or turbulent ?

- **Experimentally**

Visualization, hot wire anemometers, LDA (Laser Doppler Anemometry).

- **Numerical experiments**

Start numerical simulation selected to unsteady laminar flow. As soon as the solution converges to steady solution for sufficiently fine grid the flow regime is probably laminar

- **Re_{crit}**

According to value of Reynolds number using literature data of critical Reynolds number

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Transition Laminar-Turbulent

Stability analysis

Momentum equation for disturbance

Linear stability analysis

$$\rho\left[\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}^{(v)}}{\partial x_j}-\rho\left[u_j\frac{\partial U_i}{\partial x_j}\right]-\left\{u_j\frac{\partial u_i}{\partial x_j}-\rho\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle\right\}$$

Velocity disturbance

Mean (undisturbed) flow

Production (extracting energy from the mean flow to fluctuations)

linear stability theory can predict *when* many flows become *unstable*, it can say very little about *transition to turbulence* since this progress is highly non-linear

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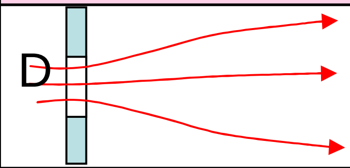
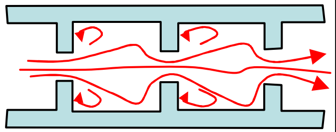
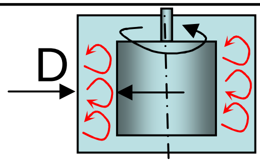
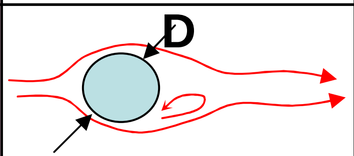
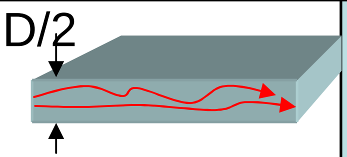
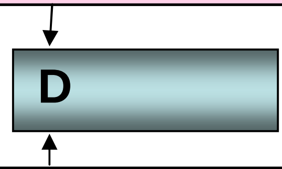
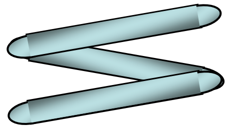
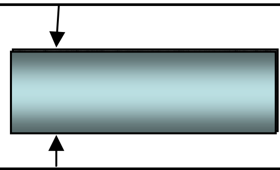
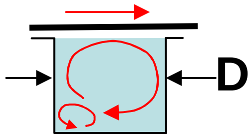
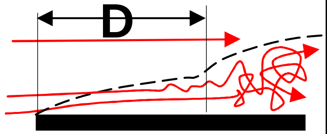
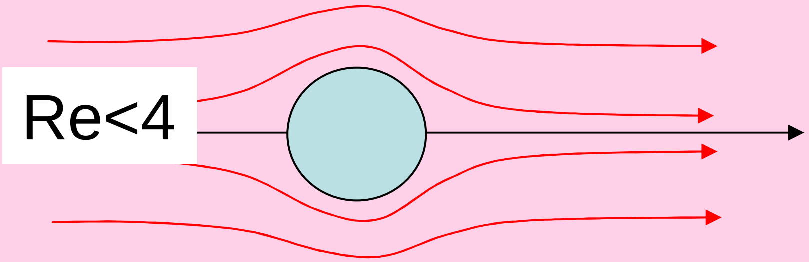
Transition Laminar-Turbulent

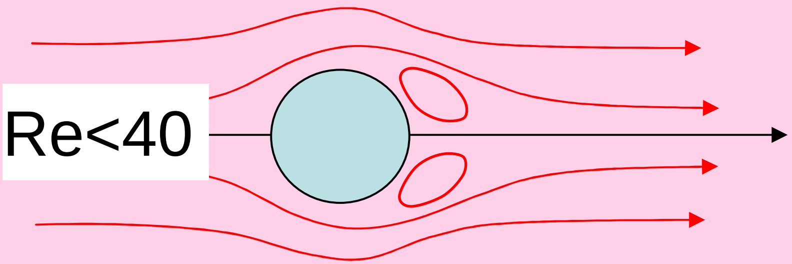
	Geometry	Re_{crit}
Jets		5-10
Baffled channels		100
Couette flow		300
Cross flow		400
Planar channel		1000

	Geometry	Re_{crit}
Circ.pipe		2000
Coiled pipe		5000
Suspension in pipe		7000
Cavity		8000
Plate		500000

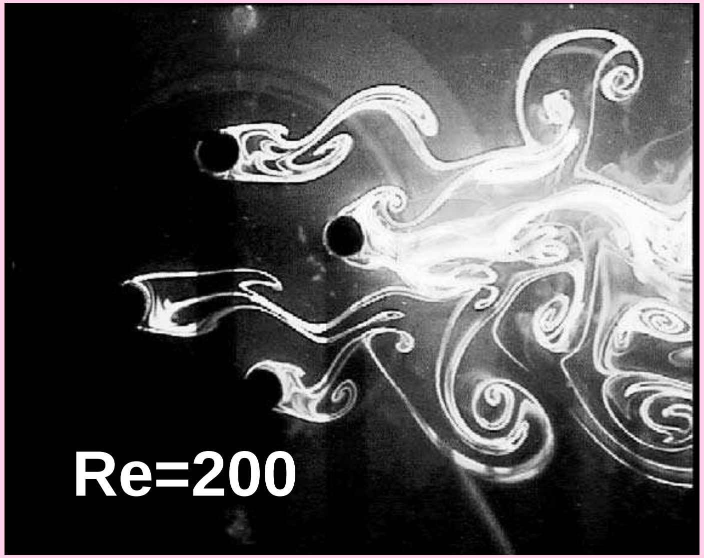
CFD6 Turbulent structures evolution

Journal of Fluids and Structures 18 (2003) 305–324
Force coefficients and Strouhal numbers of four cylinders in cross flow
K. Lama, J.Y. Lib, R.M.C. Soa





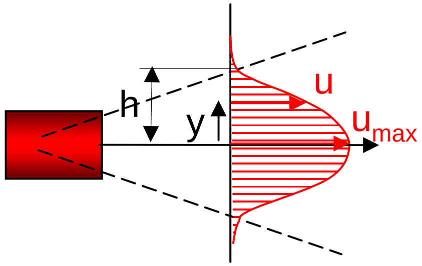
Re<200 2D von Karman vortex street

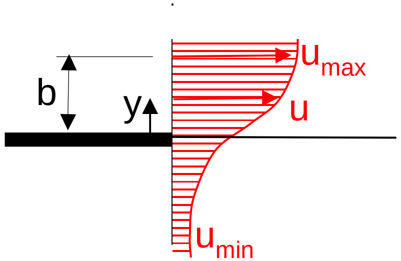


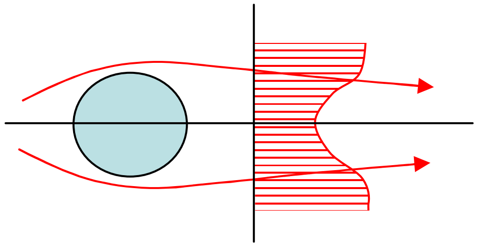
CFD6 **Fully developed turbulent flows**

Free flows (self preserving flows)

Jets | Mixing layers | Wakes



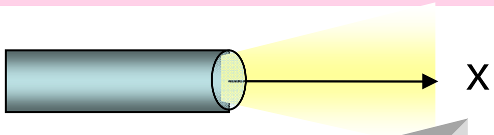




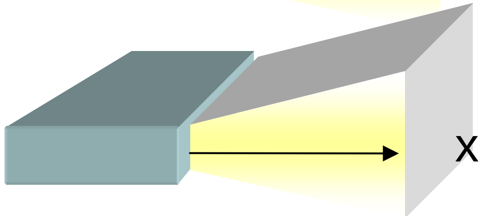
$$\frac{u}{u_{max}} = f\left(\frac{y}{h}\right) \qquad \frac{u - u_{min}}{u_{max} - u_{min}} = g\left(\frac{y}{b}\right)$$

Jet thickness ~x, mixing length ~x see Goertler, Abramovic Teorieja turbulentnych struj, Moskva 1984:

Circular jet $u_{max} = \frac{const}{x}$



Planar jet $u_{max} = \frac{const}{\sqrt{x}}$



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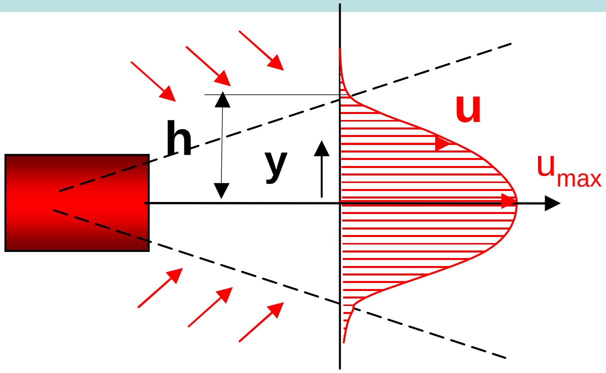
Example - tutorial

Entrainment in jets (increase of volumetric flowrate)



$$\frac{u}{u_{max}} = f\left(\frac{y}{h}\right)$$

Planar jet

~x

~1/√x

$$\dot{V}(x) = \int_0^h u_{max}(x) f\left(\frac{y}{h}\right) dy = h(x) u_{max}(x) \int_0^1 f(\eta) d\eta = h(x) u_{max}(x) F_{planar} = c\sqrt{x}$$

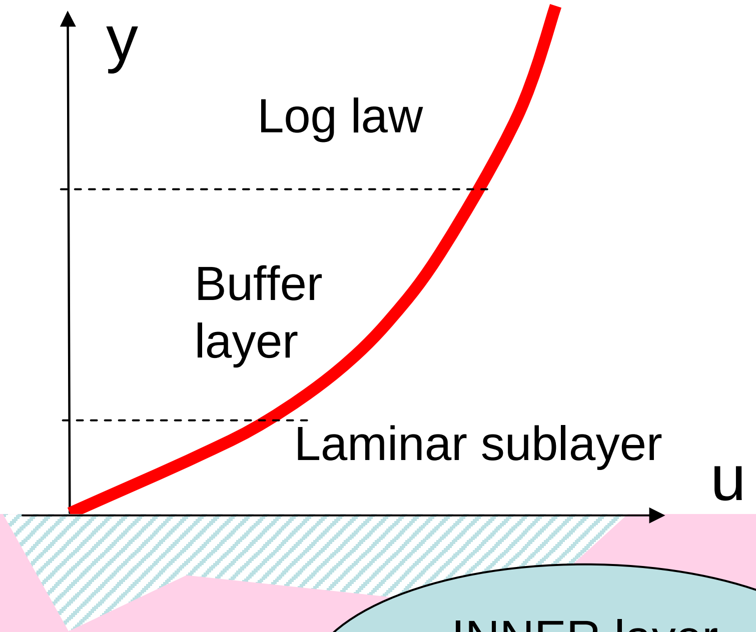
Circular jet

~1/x

$$\dot{V}(x) = 2\pi \int_0^h u_{max}(x) f\left(\frac{y}{h}\right) y dy = h^2(x) u_{max}(x) \int_0^1 \eta f(\eta) d\eta = h^2(x) u_{max}(x) F_{circ} = cx$$

CFD6 Fully developed turbulent flows

Flow at walls (boundary layers)



Friction velocity

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

$$u^+ = \frac{u}{u^*} \qquad y^+ = \frac{y u^* \rho}{\mu}$$

INNER layer (independent of bulk velocities)

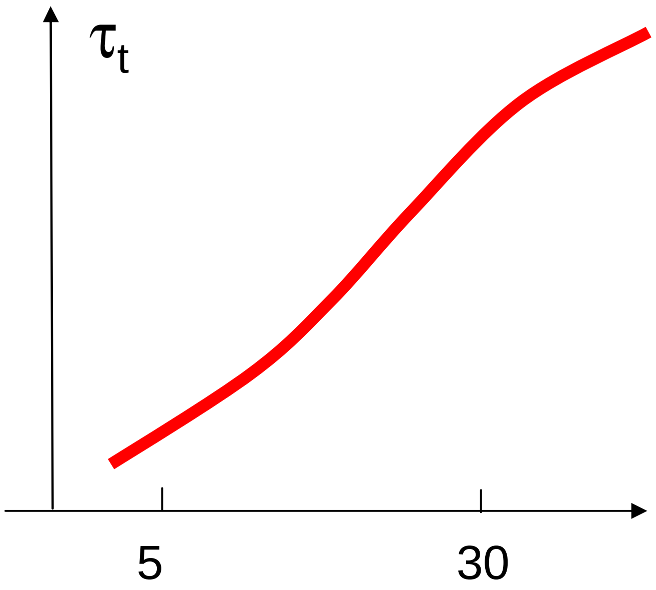
laminar	$y^+<5$	$u^+=y^+$
buffer	$5<y^+<30$	$u^+=-3+5\ln y^+$
Log law	$30<y^+<500$	$u^+=\ln Ey^+/\kappa$

OUTER layer (law of wake)

$$\frac{u_{max} - u}{u^*} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + A$$

CFD6 Fully developed turbulent flows

Flow at walls (turbulent stresses)



$$y^+ = \frac{y u^* \rho}{\mu}$$

Prandtl's model

$$\tau_t = \rho l_m^2 \left(\frac{\partial u}{\partial y}\right)^2$$

$$l_m = \kappa y [1 - \exp(-\frac{y^+}{26})]$$

vanDriest model of mixing length l_m

$$\tau_t = \rho \overline{u'v'} = 24 \kappa^2 \tau_w [1 - \exp(-\frac{y^+}{26})]^2$$

CFD6 Example tutorial

Calculate thickness of laminar sublayer at flow of water in pipe (D=2 cm) at flowrate 1 l/s.

$$\mathrm{Re} = \frac{4\dot{V}\rho}{\pi D \mu} = \frac{4 \cdot 10^{-3} \cdot 1000}{\pi 0.02 \cdot 10^{-3}} = 63662$$

Turbulent region, well within validity of Blasius correlation

Wall shear stress from Blasius

$$\tau_w = \frac{1}{8}\lambda \rho u^2 = 2\lambda\rho \frac{\dot{V}^2}{\pi^2 D^4} = 2\rho \frac{0.316 \dot{V}^2}{\sqrt[4]{\mathrm{Re}}\,\pi^2 D^4} = 25.2[Pa]$$

Friction velocity

$$u^* = \sqrt{\frac{\tau_w}{\rho}} = \frac{\sqrt{0.632}\dot{V}}{\sqrt[8]{\mathrm{Re}}\,\pi D^2} = 0.159[m/s]$$

$$5 = \frac{y u^* \rho}{\mu}$$

Dimensionless thickness of laminar sublayer

$$y = \frac{5\mu}{\rho} \frac{\pi D^2 \sqrt[8]{\mathrm{Re}}}{\dot{V}\sqrt{0.632}} = \frac{0.005}{1000} \frac{\pi 0.02^2 \sqrt[8]{63662}}{0.001\sqrt{0.632}} = \frac{\pi \cdot 2 \cdot 10^{-8} \sqrt[8]{63662}}{\sqrt{0.632}} = 0.31 \mu m$$

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TIME AVERAGING of turbulent fluctuations

RANS (Reynolds Averaging of Navier Stokes eqs.)

Time average

$$\overline{\Phi} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \Phi(t)\,dt \qquad \Phi = \overline{\Phi} + \Phi'$$

Favre's average

$$\tilde{\Phi} = \frac{\int_t^{t+\Delta t} \rho\Phi(t)\,dt}{\int_t^{t+\Delta t} \rho\,dt} \qquad \Phi = \tilde{\Phi} + \Phi''$$

$$\tilde{\Phi} = \overline{\Phi} + \frac{\overline{\rho'\Phi'}}{\overline{\rho}}$$

Proof: $\overline{(\overline{\rho} + \rho')(\overline{\Phi} + \Phi')} = \overline{\rho}\,\overline{\Phi} + \overline{\rho'\Phi'}$

Remark: Favre's average differs only for compressible substances and at high Mach number flows. Ma>1

CFD6 TIME AVERAGING of turbulent fluctuations

Trivial facts

- **Averaged value of fluctuation is zero** $\overline{\Phi'} = 0$
- **Average value of gradient of fluctuations is zero** $\overline{\dfrac{\partial \Phi'}{\partial x}} = 0$
- **Average value of product is not the product of averaged values**

$$\overline{uv} = \bar{u}\,\bar{v} + \overline{u'v'}$$

$$\overline{\nabla \bullet (\vec{u}\vec{v})} = \nabla \bullet (\bar{\vec{u}}\;\bar{\vec{v}}) + \nabla \bullet (\overline{\vec{u}'\vec{v}'})$$

CFD6 **TIME AVERAGING** **of NS equations**

Continuity equation

$$\nabla \bullet \vec{u} = 0 \qquad \overline{\nabla \bullet \vec{u}} = 0 \rightarrow \nabla \bullet \overline{\vec{u}} = 0$$

Navier Stokes equations

$$\overline{\frac{\partial}{\partial t}(\rho \vec{u})} + \overline{\nabla \bullet (\rho \vec{u}\vec{u})} = -\overline{\nabla p} + \overline{\nabla \bullet (\mu \nabla \vec{u})}$$

$$\frac{\partial}{\partial t}(\rho \overline{\vec{u}}) + \nabla \bullet (\rho \overline{\vec{u}}\ \overline{\vec{u}}) = -\nabla \overline{p} + \nabla \bullet (\mu \nabla \overline{\vec{u}}) - \nabla \bullet (\rho \overline{\vec{u}'\vec{u}'})$$

Reynolds stresses

$$\vec{\vec{\tau}}_t = -\rho \overline{\vec{u}'\vec{u}'}$$

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TIME AVERAGING of transport equations

$$\overline{\frac{\partial}{\partial t}(\rho\Phi)} + \overline{\nabla\bullet(\rho\Phi\vec{u})} = \overline{\nabla\bullet(\Gamma\nabla\Phi)}$$

$$\frac{\partial}{\partial t}(\rho\bar{\Phi}) + \nabla\bullet(\rho\bar{\vec{u}}\;\bar{\Phi}) = \nabla\bullet(\Gamma\nabla\bar{\Phi}) - \nabla\bullet(\rho\overline{\Phi'\vec{u}'})$$

Turbulent fluxes

$$\vec{q}_t = -\rho\overline{\Phi'\vec{u}'}$$

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Boussinesq hypothesis

Turbulent fluxes and turbulent stresses are defined by the same constitutive equations as in laminar flows, just only replacing diffusion coefficients and viscosity by turbulent transport coefficients.

CFD6 **Boussinesq** **hypothesis**

Analogy to Fourier law $\vec{q} = -\lambda \nabla T$

$$\rho\overline{\Phi'\vec{u}'} = -\Gamma_t \nabla\overline{\Phi}$$

$$\rho\overline{\Phi' u_i'} = -\Gamma_t \frac{\partial\overline{\Phi}}{\partial x_i}$$

Analogy to Newton's law $\vec{\vec{\tau}} = \mu(\nabla\vec{u} + (\nabla\vec{u})^T)$

$$\rho\overline{\vec{u}'\vec{u}'} = -\mu_t(\nabla\overline{\vec{u}} + (\nabla\overline{\vec{u}})^T)$$

$$\rho\overline{u_i' u_j'} = -\mu_t\left(\frac{\partial\overline{u}_j}{\partial x_i} + \frac{\partial\overline{u}_i}{\partial x_j}\right)$$

$$\vec{\vec{\tau}}_t = 2\mu_t\vec{\vec{E}} \qquad \vec{\vec{E}} = \frac{1}{2}(\nabla\overline{\vec{u}} + (\nabla\overline{\vec{u}})^T)$$

Rate of deformation based upon gradient of averaged velocities

CFD6 TRANSPORT coefficients-analogy

It is assumed that the rate of turbulent transport based upon migration of turbulent eddies is the same for momentum, mass and energy, therefore all transport coefficients should be almost the same

$$\frac{\mu_t}{\Gamma_t} = \sigma_t$$

Prandtl number for turbulent heat transfer
Schmidt number for turbulent mass transfer

σ_t=0.9 at walls

σ_t=0.5 for jets

according to Rodi

CFD6 Turbulent viscosity models

Turbulent viscosity is not a material parameter. It depends upon the actual velocity field and fluctuations at current point x,y,z. There exist different RANS models for turbulent viscosity prediction

- Algebraic models (not reflecting transport of eddies)
- 1 equation models (transport equation for turbulent viscosity)
- 2 equations models (viscosity derived from transport equations of other characteristics of turbulent eddies)
- Nonlineasr eddy viscosity models (v2-f)
- RSM Reynolds Stress Modelling (transport equations for components of reynolds stresses)

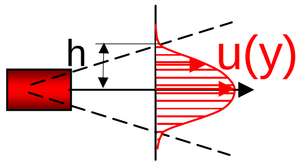
CFD6 Algebraic models

Prandtl's model of mixing length. Turbulent viscosity is derived from analogy with gases, based upon transport of momentum by molecules (kinetic theory of gases). Turbulent eddy (driven by main flow) represents a molecule, and mean path between collisions of molecules is substituted by mixing length.

Excellent model for jets, wakes, boundary layer flows. Disadvantage: fails in recirculating flows (or in flows where transport of eddies is very important).

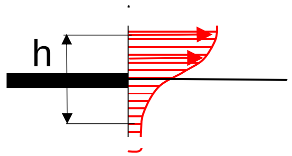
$$\mu_t = \rho l_m^2 \frac{\partial u}{\partial y}$$

Mixing length

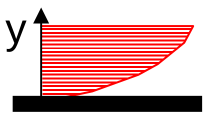


$l_m = 0.075h$ Circular jet

$l_m = 0.09h$ Planar jet



$l_m = 0.07h$ Mixing layers



$$l_m = \kappa y[1 - \exp(-\frac{y^+}{26})]$$

Currently used algebraic models are Baldwin Lomax, and Cebecci Smith

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2 equations models

Two equation models calculate turbulent viscosity from the pairs of turbulent characteristics k-ε, or k-ω, or k-l (Rotta 1986)

k (kinetic energy of turbulent fluctuations) [m²/s²]

ε (dissipation of kinetic energy) $\varepsilon \sim u^3/l$ [m²/s³]

ω (specific dissipation energy) [1/s]

$$\mu_t = \rho \frac{k}{\omega}$$

Wilcox (1998) k-omega model, Kolmogorov (1942)

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

Jones,Launder (1972), Launder Spalding (1974)
k-epsilon model

$C_\mu = 0.09$ (Fluent-default)

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Tutorial 2 equations models

Derive relationship for turbulent viscosity

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

Launder Spalding k-epsilon model

$$\mu = \rho^\alpha \varepsilon^\beta k^\gamma$$

$$[Pa.s = \frac{Ns}{m^2} = \frac{kg}{ms}] = [\frac{kg^\alpha}{m^{3\alpha}}][\frac{m^{2\beta}}{s^{3\beta}}][\frac{m^{2\gamma}}{s^{2\gamma}}] = \left[\frac{kg^\alpha m^{2\gamma+2\beta-3\alpha}}{s^{2\gamma+3\beta}}\right]$$

$$1 = \alpha$$

$$-1 = 2\gamma + 2\beta - 3$$

$$1 = 2\gamma + 3\beta$$

$$\alpha = 1$$

$$\beta = -1$$

$$\gamma = 2$$

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k – transport equation

All two equation models (and also Prandtl's one equation model) are based upon transport equation for the kinetic energy of turbulence

$$k=\frac{1}{2}\overline{u_i'u_i'}=\frac{1}{2}(\overline{u'^2}+\overline{v'^2}+\overline{w'^2})$$

Transport equation for k is derived in a similar way like the transport equation of mechanical energy by multiplying NS equation by vector of velocities (unlike mechanical energy only by the vector of velocity fluctuations)

$$\overline{\vec{u}'\bullet\frac{\partial\rho\vec{u}}{\partial t}}+\overline{\vec{u}'\bullet\nabla\bullet(\rho\vec{u}\vec{u})}=-\overline{\vec{u}'\bullet\nabla p}+\overline{\vec{u}'\bullet\nabla\bullet(\mu\nabla\vec{u})}$$

$$\overline{\vec{u}'\bullet\frac{\partial\rho\vec{u}}{\partial t}}=\frac{\partial}{\partial t}(\rho\frac{1}{2}\overline{\vec{u}'\bullet\vec{u}'})=\frac{\partial\rho k}{\partial t}$$

$$\frac{\partial\rho k}{\partial t}+\nabla\bullet(\rho k\vec{u})=\nabla\bullet(-\overline{p'\vec{u}'}+2\mu\overline{\vec{u}'\bullet\vec{\vec{e}}'}-\frac{\rho}{2}\overline{\vec{u}'\bullet\vec{u}'\vec{u}'})-2\mu\overline{\vec{\vec{e}}':\vec{\vec{e}}'}-\rho\overline{\vec{u}'\vec{u}'}:\vec{\vec{E}}$$

Transport of pressure (p') | viscous stress (2μe') | reynolds stress (ρu'u') | rate of dissipation | turbul.production

CFD6 k – transport equation

Dispersion terms (viscous and reynolds stresses) and production term can be expressed using turbulent viscosity (Boussinesq)

Example: approximation of Reynolds stress transport term

$$\overline{\vec{u}'k'} \sim \mu_t \nabla k \qquad \frac{\rho}{2}\overline{\vec{u}'\bullet\vec{u}'\vec{u}'} = \rho\overline{k'\vec{u}'} = \Gamma_k \nabla k$$

$$\frac{\partial \rho k}{\partial t} + \nabla\bullet(\rho k \vec{u}) = \nabla\bullet(\frac{\mu_t}{\sigma_k}\nabla k) + 2\mu_t \vec{\vec{E}}:\vec{\vec{E}} - 2\mu\overline{\vec{\vec{e}}':\vec{\vec{e}}'}$$

$\sigma_k = 1$ Fluent

$-\rho\varepsilon$

Rate of dissipation of kinetic energy of velocity fluctuation

Laminar and not turbulent viscosity!

$$\varepsilon = \frac{2\mu}{\rho}\overline{\vec{\vec{e}}':\vec{\vec{e}}'} = \frac{2\mu}{\rho}\overline{e_{ij}'e_{ij}'}$$

$$e_{ij}' = \frac{1}{2}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)$$

CFD6 ε – transport equation

Transport equation for dissipation ε looks like the k-transport. Just substitute ε for k. Production and dissipation terms are modified by universal constants $C_{1\varepsilon}$ and $C_{2\varepsilon}$

Production term
(generator of turbulence)

Dissipation
term

$$\frac{\partial \rho\varepsilon}{\partial t} + \nabla\bullet(\rho\varepsilon\vec{u}) = \nabla\bullet(\frac{\mu_t}{\sigma_\varepsilon}\nabla\varepsilon) + (C_{1\varepsilon}2\mu_t\vec{\vec{E}}:\vec{\vec{E}} - C_{2\varepsilon}\rho\varepsilon)\frac{\varepsilon}{k}$$

This term follows from dimensional analysis

$$[\frac{\varepsilon}{k}]\overset{!}{\equiv}[\frac{1}{s}]$$

$C_{1\varepsilon}$ = 1.44 $C_{2\varepsilon}$ = 1.92 σ_ε= 1.3 Fluent (default)

CFD6 k-ε modifications (RNG, realizable)

Corrections of dispersion, production and dissipation terms with the aim to extent applicability of k-ε for **low Reynolds number** flows

$$\frac{\partial \rho k}{\partial t}+\nabla\bullet(\rho k\vec{u})=\nabla\bullet((\mu+\frac{\mu_t}{\sigma_k})\nabla k)+2\mu_t\vec{\vec{E}}:\vec{\vec{E}}-\rho\varepsilon$$

$$\frac{\partial \rho \varepsilon}{\partial t}+\nabla\bullet(\rho \varepsilon\vec{u})=\nabla\bullet((\mu+\frac{\mu_t}{\sigma_\varepsilon})\nabla \varepsilon)+(C_{1\varepsilon}\, f_1 2\mu_t\vec{\vec{E}}:\vec{\vec{E}}-C_{2\varepsilon}\, f_2\rho\varepsilon)\frac{\varepsilon}{k}$$

Functions of turbulent Reynolds $\mathrm{Re}_t=\frac{k^2}{\varepsilon\nu}=\frac{\nu_t}{C_\mu\nu}$

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Effective viscosity (RNG)

Blending formula for effective viscosity

$$\mu_{ef} = \mu[1+\sqrt{\frac{C_\mu \rho}{\mu}}\frac{k}{\sqrt{\varepsilon}}]^2$$

$$k \rightarrow 0 \quad \text{(laminar)} \quad \mu_{ef} \rightarrow \mu$$

$$k \rightarrow \infty \quad \text{(high Re)} \quad \mu_{ef} \rightarrow \mu \frac{C_\mu \rho k^2}{\mu \varepsilon} = \mu_t$$

CFD6 ε - estimate

Dissipated energy in a mixing tank

$$F \sim \rho d^2 u^2 \sim \rho d^4 n^2$$

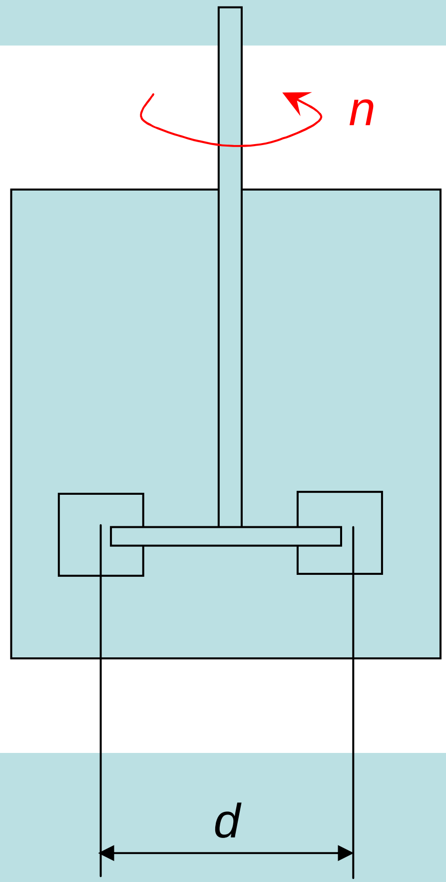
$$M_t = Fd \sim \rho d^5 n^2$$

$$P = M_t \omega \sim \rho d^5 n^3$$

Power number

$$Po = \frac{P}{\rho n^3 d^5} \approx 1$$

$$\varepsilon = \frac{P}{d^3} = Po \rho n^3 d^2$$



CFD6 k-ε boundary conditions

k and ε must be specified at inlets. Estimate of kinetic energy k is based either upon measurement (anemometers) or experience (from estimated intensity of turbulence I). Dissipation is estimated from correlations for power consumption estimates.

Values of k and ε at wall (must be also defined as boundary conditions) can be approximated by **wall functions**

$$k_w = \frac{u^{*2}}{\sqrt{C_\mu}}$$

Friction velocity $u^* = \sqrt{\frac{\tau_w}{\rho}}$

$$\varepsilon_w = \frac{u^{*3}}{\kappa y}$$

Distance of the nearest boundary node from wall

This is implemented in majority of CFD programs

CFD6 **Reynolds stresses (k-ε)**

Constitutive equation for Reynolds stresses must be modified with respect to isotropic pressure

$$\tau_{tij} = -\rho\overline{u_i'u_j'} = \mu_t\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij}$$

$$-\rho\overline{u_i'u_i'} = -2\rho k = \mu_t\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ii} = 2\mu_t\frac{\partial u_i}{\partial x_i} - 2\rho k$$

This term is zero for incompressible liquids

$$\nabla\bullet\vec{u} = 0$$

Remark: Turbulent stresses determine kinetic energy of turbulent fluctuations

CFD6

Assesment of k-ε models

Problems and erroneous results can be expected

- Unconfined flows (wakes, jets). k-ε model overestimates dissipation, therefore jets are overdumped.
- Pressure transport term (u'p') is neglected (errors in flows characterised by high pressure gradients)
- Curved boundaries or swirling flows
- Fully developed flows in noncircular ducts should be characterized by secondary flows due to anisotropy of normal Reynolds stresses (these features cannot be predicted by linear viscosity models)
- Problems in buoyancy driven flows

CFD6 Discrepancies?

Special case: Steady unidirectional flow, constant density, homogeneous turbulence

$$\cancel{\frac{\partial \rho k}{\partial t}} + \cancel{\nabla \bullet (\rho k \vec{u})} = \cancel{\nabla \bullet (\frac{\mu_t}{\sigma_k} \nabla k)} + 2\mu_t \vec{\vec{E}} : \vec{\vec{E}} - \rho\varepsilon$$

These terms are identically zero in homogeneous turbulence

$$\varepsilon = \frac{\mu_t}{\rho} (\frac{\partial u}{\partial y})^2$$

$$\cancel{\frac{\partial \rho \varepsilon}{\partial t}} + \cancel{\nabla \bullet (\rho \varepsilon \vec{u})} = \cancel{\nabla \bullet (\frac{\mu_t}{\sigma_\varepsilon} \nabla \varepsilon)} + (C_{1\varepsilon} 2\mu_t \vec{\vec{E}} : \vec{\vec{E}} - C_{2\varepsilon} \rho\varepsilon) \frac{\varepsilon}{k}$$

$$\varepsilon = \frac{\mu_t}{\rho} (\frac{\partial u}{\partial y})^2 \frac{C_{2\varepsilon}}{C_{1\varepsilon}}$$

Further reading

$C_{1\varepsilon} = 1.44$ $C_{2\varepsilon} = 1.92$ $\sigma_k = 1$ $\sigma_\varepsilon = 1.3$

CFD6 Reynolds Stress Models (RSM)

$$R_{ij} = \overline{u_i ' u_j '}$$

6 transport equations

$$\frac{DR_{ij}}{Dt} = P_{ij} + \nabla \bullet (\frac{\nu_t}{\sigma} \nabla R_{ij}) - \varepsilon_{ij} + \Pi_{ij} + \Omega_{ij}$$

Centrifugal forces

production

diffusion

Dissipation $\frac{2}{3} \varepsilon \delta_{ij}$

Pressure transport

1 dissipation

$$\frac{\partial \rho \varepsilon}{\partial t} + \nabla \bullet (\rho \varepsilon \vec{u}) = \nabla \bullet (\frac{\mu_t}{\sigma_\varepsilon} \nabla \varepsilon) + (C_{1\varepsilon} 2 \mu_t \vec{\vec{E}} : \vec{\vec{E}} - C_{2\varepsilon} \rho \varepsilon) \frac{\varepsilon}{k}$$

CFD6 Large Eddy Simulation (LES)

Instead of time averaging of NS equations, spatial fluctuation filtering of NS equations (only small eddies are removed, motion of large eddies is calculated).

Filter is realized by convolution integral

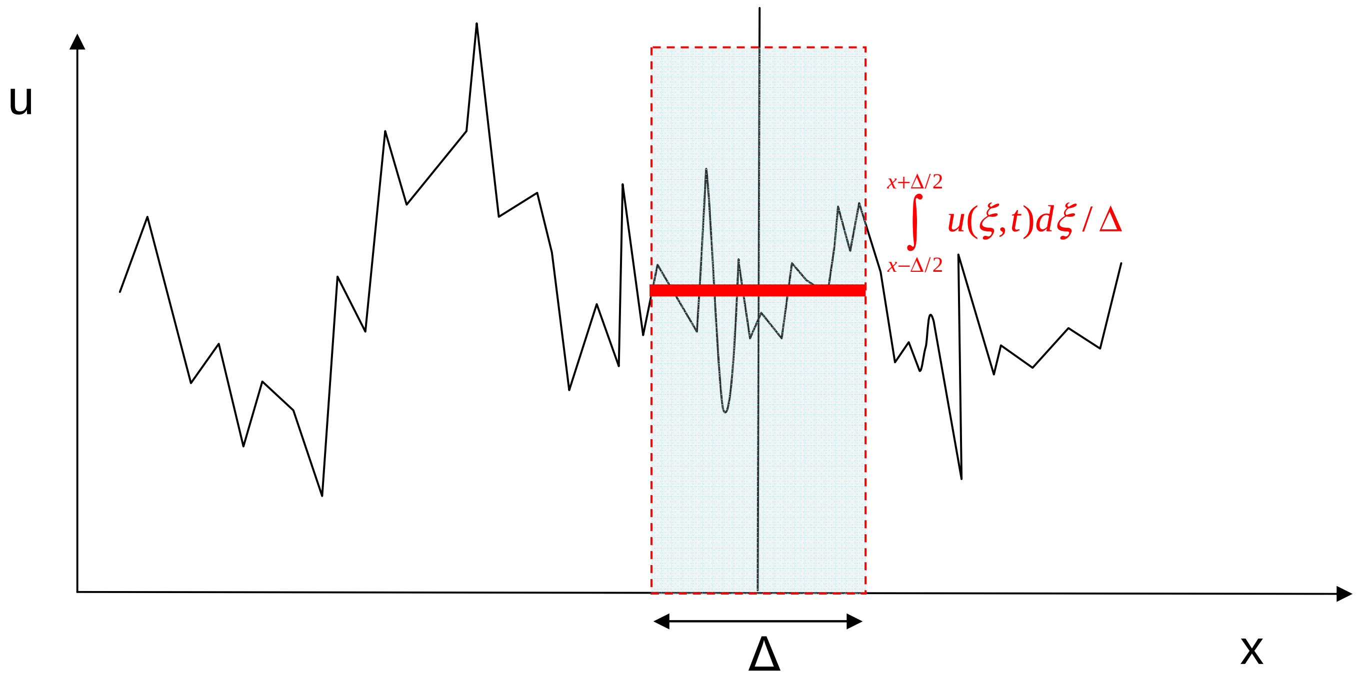
$$\overline{u}(x,t) = \iiint_{\Omega} G(x-\xi,\Delta)u(\xi,t)d\xi = \iiint_{\Omega} G(\xi,\Delta)u(x-\xi,t)d\xi$$

Convolution, Green's function G

CFD6 LES G-function example

$$G(x-\xi,\Delta) = 1/\Delta^3 \qquad \text{for} \quad |x-\xi| < \Delta/2$$

$$G(x-\xi,\Delta) = 0 \qquad \text{for} \quad |x-\xi| \geq \Delta/2$$



CFD6 LES filtering - properties

$$\bar{u} \neq \bar{\bar{u}}$$

Basic difference in comparison with time averaging

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \bar{u}}{\partial x} = \frac{\partial}{\partial x} \iiint G(\xi) u(x-\xi, t) d\xi$$

CFD6 LES NS equations

Continuity equation without changes

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i u_j})\right) = -\frac{\partial \overline{p}}{\partial x_i} + \mu \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k}$$

$$\overline{u_i u_j} = \overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')} = \overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i' \bar{u}_j} + \overline{u_i' u_j'} =$$

$$= \bar{u}_i \bar{u}_j + \underbrace{\overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j}_{L_{ij}} + \underbrace{\overline{\bar{u}_i u_j'} + \overline{u_i' \bar{u}_j}}_{C_{ij}} + \underbrace{\overline{u_i' u_j'}}_{R_{ij}}$$

L_{ij} Leonard stresses

C_{ij} cross stresses

R_{ij} SGS (sub grid stresses)

CFD6

LES NS equations

Leonard stresses are usually neglected because they are of the second order – almost the same as discretisation error. Cross and subgrid stresses are usually modeled together using turbulent viscosity approach.

$$\rho(\frac{\partial \overline{u_i}}{\partial t}+\frac{\partial}{\partial x_j}(\overline{u}_i\overline{u}_j))=-\frac{\partial \overline{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\mu\frac{\partial \overline{u_i}}{\partial x_j}+\tau_{ij})$$

It looks like Reynolds stresses

CFD6 LES Smagorinski SGS

Smagorinski model of subgrid stresses is almost the same as the Prandtl's model of mixing length, only instead of the mixing length is substituted a filter size.

$$\tau_{ij} = 2\mu_T E_{ij}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

$$\mu_T = \rho (C_s \Delta)^2 \sqrt{\vec{\vec{E}} : \vec{\vec{E}}}$$